

# A complete radical formula and 2-primal modules

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## Abstract

We introduce a complete radical formula for modules over non-commutative rings which is the equivalence of a radical formula in the setting of modules defined over commutative rings. This gives a general frame work through which known results about modules over commutative rings that satisfy the radical formula are retrieved. Examples and properties of modules that satisfy the complete radical formula are given. For instance, it is shown that a module that satisfies the complete radical formula is completely semiprime if and only if it is a subdirect product of completely prime modules. This generalizes a ring theoretical result: a ring is reduced if and only if it is a subdirect product of domains. We settle in affirmative a conjecture by Groenewald and the current author given in [5] that a module over a 2-primal ring is 2-primal. More instances where 2-primal modules behave like modules over commutative rings are given. This is in tandem with the behaviour of 2-primal rings of exhibiting tendencies of commutative rings. We end with some questions about the role of 2-primal rings in algebraic geometry.

**Keywords:** 2-primal rings, 2-primal modules, modules that satisfy the radical formula, modules that satisfy the complete radical formula.

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## 1 Introduction

A proper ideal  $\mathcal{P}$  of a ring  $R$  is a *prime* (resp. *completely prime*) ideal if for all ideals  $\mathcal{A}, \mathcal{B}$  (resp. elements  $a, b$ ) of  $R$  such that  $\mathcal{A}\mathcal{B} \subseteq \mathcal{P}$  (resp.  $ab \in \mathcal{P}$ ), we have either  $\mathcal{A} \subseteq \mathcal{P}$  or  $\mathcal{B} \subseteq \mathcal{P}$  (resp.  $a \in \mathcal{P}$  or  $b \in \mathcal{P}$ ). The prime radical (resp. completely prime radical) of a ring  $R$  is the intersection of all prime (resp. completely prime) ideals of  $R$ . Let  $\mathcal{N}(R)$ ,  $\beta(R)$  and  $\beta_{co}(R)$  denote the set of all nilpotent elements of  $R$ , the prime radical of  $R$  and the completely prime radical of  $R$  (also called the generalized nilradical of  $R$ ). Let  $N$  be a submodule of an  $R$ -module  $M$ . The envelope of a submodule  $N$  of an  $R$ -module  $M$  is the set

$$E_M(N) := \{rm : r \in R, m \in M \text{ and } r^k m \in N \text{ for some } k \in \mathbb{N}\}.$$

In general,  $E_M(N)$  is not a submodule of  $M$ . We denote a submodule of  $M$  generated by the envelope of  $N$  by  $\langle E_M(N) \rangle$ . The elements of  $\langle E_M(0) \rangle$  are called nilpotent elements of  $M$ .

If  $R$  is a commutative ring (or a 2-primal ring), then

$$E_R(0) = \mathcal{N}(R) = \beta(R) = \beta_{co}(R). \quad (1)$$

A desire to have Equation (1) (or parts of Equation (1)) in the module setting, forms the basis of our study in this paper. In literature, there are studies about the equivalences  $\langle E_M(N) \rangle = \beta^s(N)$  and  $\beta(M) = \beta_{co}(M)$  in which case one says the submodule  $N$  of a module  $M$  satisfies the radical formula and a module  $M$  is 2-primal respectively.  $\beta^s(N)$  denotes the intersection of all prime submodules of a module  $M$  containing a submodule  $N$  and  $\beta(M)$  (resp.  $\beta_{co}(M)$ ) denotes the intersection of all prime (resp. completely prime) submodules of a module  $M$ . Whereas we supplement studies of these two equivalences, we also introduce submodules that satisfy the complete radical formula, i.e., submodules  $N$  of modules  $M$  for which  $\langle E_M(N) \rangle = \beta_{co}^s(N)$ , where  $\beta_{co}^s(N)$  is the intersection of all completely prime submodules of a module  $M$  containing a submodule  $N$ . This, naturally generalizes the notion of submodules that satisfy the radical formula in modules over commutative rings to modules over non-commutative rings. For an arbitrary ring, Levitzki showed that the set of all strongly nilpotent elements coincides with the prime radical of that ring. We give examples of modules that satisfy the module analogue of Levitzki result.

All rings are unital and associative. The modules are left modules defined over rings.

## Paper road map

This paper has seven sections. We give the introduction in Section 1. Preliminary results which are needed later in the sequel are given in Sections 2 and 3. Section 2 focuses on 2-primal rings and some of their properties, whereas in Section 3 we give relevant information about module analogues of well known notions in ring theory. They include: prime modules, completely prime modules, modules that satisfy the radical formula and 2-primal modules. It is in Section 4 that we introduce the complete radical formula of modules. As examples, it is shown that the following modules satisfy the complete radical formula: a projective and 2-primal module [Theorem 4.8], a finitely generated module over a 2-primal ring [Theorem 4.9], a completely prime module and the regular module  ${}_R R$  when  $R$  is a 2-primal ring [Theorem 4.14]. In Corollaries 4.6 and 4.7, we have given several modules which are projective and 2-primal. Furthermore, all the modules given above also satisfy both the radical formula as well as the module analogue of Levitzki result for rings. A ring  $R$  satisfies a complete radical formula if every  $R$ -module satisfies the complete radical formula. It is shown that every semisimple 2-primal ring satisfies the complete radical formula [Corollary 4.13]. We give a new characterization of 2-primal

rings. A ring  $R$  is 2-primal if and only if  $\beta(R) = E_R(0)$  [Theorem 4.2]. In Section 5, we give an application of modules that satisfy the complete radical formula. If a module satisfies the complete radical formula, then it is completely semiprime if and only if it is a subdirect product of completely prime modules [Theorem 5.1]. In Section 6, we prove in affirmative a conjecture posed in [5]; it states that a module over a 2-primal ring is 2-primal. In Section 7, which is the last section, we give some information which inhibits the use of non-commutative rings in algebraic geometry. However, given the behaviour of 2-primal rings, i.e., having behavioural tendencies of commutative rings, we pose some questions on possible of using 2-primal rings in algebraic geometry.

## 2 2-primal rings

**Definition 2.1.** *A ring  $R$  is 2-primal if*

$$\mathcal{N}(R) = \beta(R).$$

All commutative rings and all reduced rings are 2-primal. The class of 2-primal rings has been widely studied, see for instance [2, 14, 15, 16, 21] among others.

**Proposition 1.** [2, Proposition 2.1] *Let  $R$  be a ring, the following statements are equivalent:*

1.  $R$  is 2-primal,
2.  $\beta_{co}(R) = \beta(R)$ .

An ideal  $\mathcal{I}$  of a ring  $R$  is 2-primal if

$$\beta_{co}(R/\mathcal{I}) = \beta(R/\mathcal{I}). \tag{2}$$

It follows that a ring is 2-primal if and only if its zero ideal is 2-primal. Equality (2) is the basis for the definition of 2-primal submodules, see Definition 3.5.

The class of 2-primal rings is large. It contains many classes of generalizations of commutative rings: symmetric rings, IFP/SI rings, reversible rings, PSI rings, semi-symmetric rings, etc. For examples and chart of implications among these classes, see [2] and [14].

2-primal rings behave like commutative rings. For instance, just like commutative rings, they possess the following properties:

1. their sets of all nilpotent elements are ideals;
2. they are Dedekind finite, i.e., if  $R$  is a 2-primal ring and  $a, b \in R$  such that  $ab = 1$ , then  $ba = 1$ ;

3. if  $R$  is a 2-primal ring, then the ring  $R/\beta(R)$  is reduced, and hence it is IFP (i.e., if  $a, b \in R$ , then  $ab \in \beta(R)$  implies that  $aRb \subseteq \beta(R)$ ), reversible (i.e., if  $a, b \in R$ , then  $ab \in \beta(R)$  implies that  $ba \in \beta(R)$ ) and symmetric (i.e., if  $a, b, c \in R$ , then  $abc \in \beta(R)$  implies that  $acb \in \beta(R)$ );
4. they satisfy Kothe conjecture, i.e., the sum of two nil left ideals is nil;
5. prime ideals are completely prime and hence are strongly prime, strictly prime,  $l$ -prime and  $s$ -prime;
6. they cannot be full matrix rings, [14, p. 495];
7. the equality

$$\mathcal{N}_s(R) = E_R(0) = \mathcal{N}(R) = \beta(R) = \beta_{co}(R)$$

holds, where  $\mathcal{N}_s(R)$  is the set of all strongly nilpotent elements of  $R$ , see Corollary 4.3;

8. semisimple 2-primal rings satisfy the radical formula, see Corollaries 4.12 and 4.13, and Proposition 8.

**Definition 2.2.** [25, Definition 19], [8, p. 742] *A filtered ring  $A$  is said to be almost commutative if the associated graded ring,  $gr A = \bigoplus_{i \in I} (A_i/A_{i-1})$  is commutative.*

Basic examples of almost commutative rings involve rings of differential operators and universal enveloping algebras.

**Example 1.** *The almost commutative rings: the universal enveloping algebra of any Lie algebra over a field and the ring of differential operators are reduced rings and hence 2-primal.*

We are then led to ask the following question:

**Question 2.1.** *Is every almost commutative ring 2-primal?*

### 3 The module analogues

In this section, we introduce module analogues of prime rings, completely prime rings (domains), 2-primal rings and modules that mimic the equivalence  $\mathcal{N}(R) = \beta(R)$  in commutative rings.

### 3.1 Prime and completely prime modules

**Definition 3.1.** [4] A submodule  $P$  of an  $R$ -module  $M$  is a prime submodule if  $RM \not\subseteq P$  and for all ideals  $\mathcal{A}$  of  $R$  and submodules  $N$  of  $M$  such that  $\mathcal{A}N \subseteq P$ , we have  $N \subseteq P$  or  $AM \subseteq P$ .

**Definition 3.2.** [6, Definition 2.1] A submodule  $P$  of an  $R$ -module  $M$  is a completely prime submodule if  $RM \not\subseteq P$  and for all elements  $r \in R$  and  $m \in M$  such that  $rm \in P$ , we have  $m \in P$  or  $rM \subseteq P$ .

**Definition 3.3.** A proper submodule  $P$  of an  $R$ -module  $M$  is a semiprime (resp. completely semiprime) submodule of  $M$ , if  $RM \not\subseteq P$  and for all  $a \in R$  and  $m \in M$ ,  $aRam \subseteq P$  (resp.  $a^2m \in P$ ) implies that  $am \in P$ .

A module is prime (resp. completely prime, semiprime, completely semiprime) if its zero submodule is a prime (resp. completely prime, semiprime, completely semiprime) submodule. The prime (resp. completely prime) radical of a submodule  $N$  of an  $R$ -module  $M$  is the intersection of all prime (resp. completely prime) submodules of  $M$  containing  $N$ . We denote the prime (resp. completely prime) radical of a nonzero submodule  $N$  by  $\beta^s(N)$  (resp.  $\beta_{co}^s(N)$ ). Otherwise, if  $N = 0$ , we write  $\beta(M)$  (resp.  $\beta_{co}(M)$ ) and call  $\beta(M)$  (resp.  $\beta_{co}(M)$ ) the prime (resp. completely prime) radical of  $M$ . If  $M$  has no prime (resp. completely prime) submodules, we write  $\beta(M) = M$  (resp.  $\beta_{co}(M) = M$ ).

A ring  $R$  is prime (resp. completely prime, semiprime, completely semiprime) if and only if the  $R$ -module  $R$  is (resp. completely prime, semiprime, completely semiprime). Any completely prime submodule is prime. Every maximal submodule is a prime submodule but it need not be completely prime. A torsion-free module, a simple module which is Lee-Zhou reduced and a projective module over a domain are completely prime modules, see [6, Examples 2.2], [6, Example 2.3] and [23, Example 3.10]. An indecomposable projective module over a hereditary Artin algebra is a completely prime module. To see this, if  $M$  is an indecomposable projective module over a hereditary Artin algebra  $R$ , then by [7, Proposition 5.1.1] every nonzero map  $f \in \text{End}_R(M)$  is a monomorphism. It follows from [23, Proposition 3.1] that  $M$  is a completely prime module.

Let  $N$  be a submodule of an  $R$ -module  $M$ , by  $(N : M)$  we denote the ideal

$$\{r \in R : rM \subseteq N\}$$

of  $R$  which is the annihilator of the factor  $R$ -module  $M/N$ .

**Proposition 2.** A submodule  $N$  of an  $R$ -module  $M$  is completely prime (resp. prime) if and only if  $P = (N : M)$  is a completely prime (resp. prime) ideal of  $R$  and the  $R/P$ -module  $M/N$  is torsion-free.

## 3.2 The radical formula of modules

The equality  $E_R(0) = \beta(R)$  from Equation (1) for commutative rings, motivated McCasland and Moore in [17] to introduce (sub)modules that satisfy the radical formula. On the other hand, the equality  $\beta(R) = \beta_{co}(R)$  for 2-primal rings motivated Groenewald and the current author in [5] to define 2-primal modules. In this subsection, we define, give examples and compare these two types of modules.

**Definition 3.4.** *A submodule  $N$  of an  $R$ -module  $M$  satisfies the radical formula if*

$$\langle E_M(N) \rangle = \beta^s(N).$$

*A module  $M$  satisfies the radical formula if every submodule of  $M$  satisfies the radical formula. A ring  $R$  satisfies the radical formula if every  $R$ -module satisfies the radical formula.*

Modules and rings that satisfy the radical formula have been widely studied, see [3, 9, 12, 13, 17, 18, 19, 20] among others. A projective module over a commutative ring [9, Corollary 8], a module over a Dedekind integral domain [9, Theorem 9] and a representable module (and hence an Artinian module) over a commutative ring satisfy the radical formula [19, Theorem 9]. A semisimple commutative ring and an Artinian commutative ring [20] satisfy the radical formula. Not all modules defined over commutative rings satisfy the radical formula.

## 3.3 2-primal modules

**Definition 3.5.** *A submodule  $N$  of an  $R$ -module  $M$  is 2-primal if*

$$\beta(M/N) = \beta_{co}(M/N).$$

*A module  $M$  is 2-primal if its zero submodule is 2-primal, i.e., if*

$$\beta(M) = \beta_{co}(M).$$

**Proposition 3.** [5, Proposition 2.1] *A ring  $R$  is 2-primal if and only if the module  ${}_R R$  is 2-primal.*

**Example 2.** *A Lee-Zhou reduced module (see [11]) and hence a completely prime module is 2-primal. Projective modules over 2-primal rings, IFP modules, symmetric modules and modules over commutative rings are 2-primal, see [5].*

**Proposition 4.** *If the prime radical of a module  $M$  is a completely prime submodule of  $M$ , then  $M$  is a 2-primal module.*

**Proof:** Since for any completely prime submodule  $P$  of  $M$ ,  $\beta(M) \subseteq \beta_{co}(M) \subseteq P$  when  $\beta(M)$  is a completely prime submodule of  $M$ , we get  $\beta_{co}(M) \subseteq \beta(M)$  such that  $\beta_{co}(M) = \beta(M)$ . ■

We observe that “2-primal modules” is a better generalisation than “modules that satisfy the radical formula”. This is because, all modules over commutative rings are 2-primal just like all commutative rings are 2-primal. On the contrary, not all modules over commutative rings satisfy the radical formula.

There was considerable effort aimed at getting examples of modules that satisfy the radical formula. Now that there is a generalisation better than the notion of modules that satisfy the radical formula, i.e., that of 2-primal modules, it is hoped that there will be interest by different researchers to search for more examples of 2-primal modules, in addition to those pointed out in this paper and in [5].

## 4 The complete radical formula

The inequality

$$\beta_{co}(M) \subseteq \langle E_M(0) \rangle \quad (3)$$

which is equivalent to saying that  $\langle E_M(0) \rangle = \beta_{co}(M)$ , (see Lemma 1) is a necessary and sufficient condition for a zero submodule of a module  $M$  to satisfy the radical formula if and only if  $M$  is 2-primal, see [24, Corollary 2.21].

The motivation for studying (sub)modules that satisfy the complete radical formula is three fold. Firstly, it generalizes the notion of modules over commutative rings that satisfy the radical formula to modules over not necessarily commutative rings. Secondly, it is a necessary and sufficient condition for modules to be 2-primal if and only if their zero submodules satisfy the radical formula. Lastly, it allows every completely semiprime submodule to be an intersection of completely prime submodules which is not true in general.

**Definition 4.1.** *Let  $R$  be a ring and  $M$  an  $R$ -module. A submodule  $N$  of an  $R$ -module  $M$  satisfies a complete radical formula if*

$$\langle E_M(N) \rangle = \beta_{co}^s(N).$$

*A module satisfies the complete radical formula if every submodule of  $M$  satisfies the complete radical formula. A ring  $R$  satisfies the complete radical formula if every  $R$ -module satisfies the complete radical formula.*

**Lemma 1.** [24, Lemma 2.1] *If  $N$  is a submodule of an  $R$ -module  $M$ , then*

$$\langle E_M(N) \rangle \subseteq \beta_{co}^s(N).$$

**Corollary 4.1.** *If  $I$  is a left ideal of a ring  $R$ , then*

$$\langle E_R(I) \rangle \subseteq \beta_{co}^s(I).$$

We now give a new characterization of 2-primal rings.

**Theorem 4.2.** *A ring  $R$  is 2-primal if and only if*

$$\beta(R) = E_R(0).$$

**Proof:** For any ring  $R$ , it is easy to see that

$$\beta(R) \subseteq \mathcal{N}(R) \subseteq E_R(0) \subseteq \beta_{co}(R).$$

If  $R$  is 2-primal,  $\beta(R) = \beta_{co}(R)$  and hence  $\beta(R) = E_R(0)$ . For the converse,  $\beta(R) = E_R(0)$  implies that  $\beta(R) = \mathcal{N}(R)$ , which shows that  $R$  is 2-primal. ■

**Corollary 4.3.** *If  $R$  is a 2-primal ring, then*

$$\beta(R) = \mathcal{N}(R) = E_R(0) = \beta_{co}(R).$$

**Proof:** Follows from the fact that for any ring  $R$ ,  $\beta(R) \subseteq \mathcal{N}(R) \subseteq E_R(0) \subseteq \beta_{co}(R)$  and for 2-primal rings,  $\beta(R) = \beta_{co}(R)$ . ■

**Proposition 5.** *If  $N$  is a submodule of an  $R$ -module  $M$ , then the following statements are equivalent:*

1.  $E_M(N) = N$ ,
2.  $N$  is a completely semiprime submodule of  $M$ .

**Proof:** Elementary. ■

**Corollary 4.4.** *A submodule  $N$  of an  $R$ -module  $M$  is Lee-Zhou reduced if and only if it is both IFP and satisfies  $E_M(N) = N$ .*

Corollary 4.4 allows us to paraphrase Question 2.1 posed in paper [24] as:

**Question 4.1.** *Is there a prime (resp. semiprime) module which is not completely prime (resp. Lee-Zhou reduced) but it is completely semiprime?*

A positive answer to this question would lead to an example of a module which satisfies the radical formula but not 2-primal.

An element  $m$  of an  $R$ -module  $M$  is *strongly nilpotent* [1] if  $m = \sum_{i=1}^r a_i m_i$  for some  $a_i \in R$ ,  $m_i \in M$  and  $r \in \mathbb{N}$ , such that for every  $i$  ( $1 \leq i \leq r$ ) and every sequence  $a_{i1}, a_{i2}, a_{i3}, \dots$  where  $a_{i1} = a_i$  and  $a_{in+1} \in a_{in} R a_{in}$  (for all  $n$ ), we have  $a_{ik} R m_i = 0$  for some  $k \in \mathbb{N}$ . The set of all strongly nilpotent elements of a module  $M$  is a submodule and is denoted by  $\mathcal{N}_s(M)$ .



**Lemma 2.** *For any  $R$ -module  $M$ , the following inequalities hold:*

$$\mathcal{N}_s(M) \subseteq \langle E_M(0) \rangle \subseteq \beta_{co}(M).$$

**Proof:** Let  $m \in \mathcal{N}_s(M)$ , then  $m = \sum_{i=1}^r a_i m_i$  for some  $a_i \in R$ ,  $m_i \in M$  and  $r \in \mathbb{N}$  such that for every  $i$  ( $1 \leq i \leq r$ ) and every sequence  $a_{1i}, a_{2i}, a_{3i}, \dots$  where  $a_{i1} = a_i$  and  $a_{n+1i} \in a_{ni} R a_{ni}$  for all  $n$ , we have  $a_{ki} R m_i = 0$  for some  $k \in \mathbb{N}$ . For some  $i$ , choose the sequence

$$a_i, a_i^2, a_i^4, a_i^8, \dots = \{a_i^{2^{r-1}}\}_{r=1}^\infty$$

then  $a_{1i} = a_i$  and  $a_{n+1i} \in a_{ni} R a_{ni}$  for all  $n$ . By hypothesis, there exists  $k \in \mathbb{N}$  such that  $a_{ki} R m_i = 0$ . Since  $a_{ki} = a_i^{2^{k-1}}$ , it follows that  $a_i^{2^{k-1}} m_i = 0$ . This implies  $a_i m_i \in E_M(0)$  so that  $m = \sum_{i=1}^r a_i m_i \in \langle E_M(0) \rangle$ . The second inequality follows from Lemma 1. ■

If  $R$  is a 2-primal ring, then we know that

$$\mathcal{N}(R) = \mathcal{N}_s(R) = E_R(0) = \beta_{co}(R) = \beta(R). \quad (4)$$

For modules, we have Theorem 4.5 below.

**Theorem 4.5.** *If  $M$  is a projective and 2-primal  $R$ -module, then*

$$\mathcal{N}_s(M) = \langle E_M(0) \rangle = \beta_{co}(M) = \beta(M). \quad (5)$$

*Hence, the zero submodule of  $M$  satisfies both the complete radical formula as well as the radical formula; and  $M$  satisfies the module analogue of Levitzki result for rings.*

**Proof:** If  $M$  is a projective and 2-primal module, then  $\mathcal{N}_s(M) = \beta_{co}(M) = \beta(M)$ , by [1, Theorem 3.8] and the definition of 2-primal modules. Apply Lemma 2 to complete the proof. ■

**Corollary 4.6.** *For a projective module  $M$  over any one of the following rings: reduced rings, commutative rings, left-duo rings, symmetric rings, reversible rings, IFP rings, PSI rings, semi-symmetric rings and 2-primal rings; the equality*

$$\mathcal{N}_s(M) = \langle E_M(0) \rangle = \beta_{co}(M) = \beta(M)$$

*holds. Hence, the zero submodule of  $M$  satisfies both the complete radical formula as well as the radical formula; and  $M$  satisfies the module analogue of Levitzki result for rings.*

**Proof:** Any of the above mentioned rings is 2-primal, see a chart of implications in [14]. By [5, Corollary 2.1], a projective module over a 2-primal ring is 2-primal. The rest follows from Theorem 4.5. ■

**Definition 4.2.** *An  $R$ -module  $M$  is*

1. *Lee-Zhou reduced [11] if for all  $a \in R$  and every  $m \in M$ ,  $am = 0$  implies that  $Rm \cap aM = 0$ . This is equivalent to saying that: for all  $a \in R$  and every  $m \in M$ ,  $a^2m = 0$  implies that  $aRm = 0$ ;*
2. *symmetric if for  $a, b \in R$  and  $m \in M$ ,  $abm = 0$  implies that  $bam = 0$ ;*
3. *semi-symmetric if for all  $a \in R$  and every  $m \in M$ ,  $a^2m = 0$  implies that  $(a)^2m = 0$  where  $(a)$  is the ideal of  $R$  generated by  $a \in R$ ;*
4. *IFP (i.e., it has the insertion-of-factor-property) if whenever  $am = 0$  for  $a \in R$  and  $m \in M$ , we have  $aRm = 0$ .*

**Corollary 4.7.** *For each of the following modules:*

1.  *$M$  is 2-primal and free,*
2.  *$M$  is semi-symmetric and free,*
3.  *$M$  is semi-symmetric and projective,*
4.  *$M$  is IFP and projective,*
5.  *$M$  is IFP and free,*
6.  *$M$  is symmetric and projective,*
7.  *$M$  is symmetric and free,*
8.  *$M$  is reduced and projective,*
9.  *$M$  is reduced and free,*
10.  *$R$  is commutative and  $M$  is projective,*
11.  *$R$  is commutative and  $M$  is free;*

*the equality*

$$\mathcal{N}_s(M) = \langle E_M(0) \rangle = \beta_{co}(M) = \beta(M)$$

*holds. Hence, the zero submodule of  $M$  satisfies both the complete radical formula as well as the radical formula; and  $M$  satisfies the module analogue of Levitzki result for rings.*

**Proof:** By [5, Theorems 2.2 and 2.3], and the fact that every free module is projective, each of these modules is 2-primal and projective. The rest follows from Theorem 4.5. ■

Lemma 3 below can be proved with appropriate modification of methods used to prove [17, Theorem 1.5] for modules over commutative rings.

**Lemma 3.** *Let  $\phi : M \rightarrow M'$  be an  $R$ -module epimorphism and let  $N$  be a submodule of  $M$  such that  $N \supseteq \text{Ker } \phi$ .*

- (i) *If  $\beta_{co}^s(N) = \langle E_M(N) \rangle$ , then  $\beta_{co}^s(\phi(N)) = \langle E_{M'}(\phi(N)) \rangle$ ;*
- (ii) *If  $N'$  is a submodule of  $M'$  and  $\beta_{co}^s(N') = \langle E_{M'}(N') \rangle$ , then  $\beta_{co}^s(\phi^{-1}(N')) = \langle E_M(\phi^{-1}(N')) \rangle$ .*
- (iii) *If  $\beta^s(N) = \langle E_M(N) \rangle$ , then  $\beta^s(\phi(N)) = \langle E_{M'}(\phi(N)) \rangle$ ;*
- (iv) *If  $N'$  is a submodule of  $M'$  and  $\beta^s(N') = \langle E_{M'}(N') \rangle$ , then  $\beta^s(\phi^{-1}(N')) = \langle E_M(\phi^{-1}(N')) \rangle$ .*

**Theorem 4.8.** *If the  $R$ -module  $M$  is any one of the modules given in Theorem 4.5 and Corollaries 4.6 and 4.7, then  $M$  satisfies both the complete radical formula as well as the radical formula.*

*Proof.* Let  $N$  be a submodule of  $M$ . The modules  $M$  given in Theorem 4.5 and Corollaries 4.6 and 4.7 are 2-primal and projective. Hence,  $\beta_{co}(M) = \langle E_M(0) \rangle$ . When we apply Lemma 3, by letting  $M' = M/N$  and  $N' = N$ , we get  $\beta_{co}^s(N) = \langle E_{M/N}(N) \rangle = \langle E_M(N) \rangle$ , i.e., every submodule of  $M$  satisfies the complete radical formula. A similar argument starting with  $\beta(M) = \langle E_M(0) \rangle$  shows that every submodule of  $M$  satisfies the radical formula.  $\square$

Note that Theorem 4.8 retrieves the well known result that a projective module over a commutative ring satisfies the radical formula, see [9, Corollary 8].

**Theorem 4.9.** *A finitely generated (and hence a cyclic) module over a 2-primal ring satisfies both the radical formula as well as the complete radical formula. Hence, it is 2-primal, satisfies the module analogue of Levitzki result for rings and equality (5) holds.*

**Proof:** Let  $R$  be a 2-primal ring. Since 2-primal rings are closed under direct sums (see [2]), the ring  $R^n$  for some  $n \in \mathbb{N}$  is also 2-primal. By Corollary 4.3,  $R^n$  considered as an  $R$ -module satisfies both the complete radical formula as well as the radical formula. By Lemma 3, every homomorphic image of a module that satisfies the (complete) radical formula also satisfies the (complete) radical formula. Since a finitely generated  $R$ -module is a homomorphic image of  $R^n$ , it must also satisfy the (complete) radical formula.  $\blacksquare$

Theorem 4.9 retrieves an already known result: a finitely generated module over a principal ideal domain (resp. over a Dedekind domain) satisfies the radical formula, see [17, Theorem 2] (resp. [9, Theorem 9]).

For rings, every semiprime (resp. completely semiprime) ideal of  $R$  is an intersection of prime (resp. completely prime) ideals. For modules, this is not true in general, see [9, p. 3600]. However, for modules that satisfy the complete radical formula, we have Proposition 6.

**Proposition 6.** *An  $R$ -module  $M$  satisfies the complete radical formula if and only if every completely semiprime submodule  $N$  of  $M$  is an intersection of completely prime submodules of  $M$ .*

**Proof:** Suppose  $\langle E_M(N) \rangle = \beta_{co}^s(N)$  for every submodule  $N$  of  $M$ . If  $K$  is a completely semiprime submodule of  $M$ , then by Proposition 5,  $\langle E_M(K) \rangle = K$  such that by hypothesis,  $\beta_{co}^s(K) = K$ . This shows that  $K$  is an intersection of completely prime submodules of  $M$ . Conversely, suppose that  $K$  is an intersection of completely prime submodules of  $M$ . Then  $K$  is a completely semiprime submodule of  $M$ . By Proposition 5,  $\langle E_M(K) \rangle = K$ . It follows that  $K = \langle E_M(K) \rangle \subseteq \beta_{co}^s(K) \subseteq K$  and hence  $\langle E_M(K) \rangle = \beta_{co}^s(K)$ . ■

The property of the module being 2-primal allows the module to behave as though it is defined over a commutative ring. For modules over commutative rings, there is no distinction between modules that satisfy the radical formula and those that satisfy the complete radical formula. For 2-primal modules, we have Proposition 7.

**Proposition 7.** *If  $M$  is a 2-primal module, then the following statements are equivalent:*

1.  $M$  satisfies the complete radical formula,
2.  $M$  satisfies the radical formula.

**Proof:** If  $\beta(M) = \beta_{co}(M)$ , then  $\langle E_M(0) \rangle = \beta(M)$  if and only if  $\langle E_M(0) \rangle = \beta_{co}(M)$ . ■

**Definition 4.3.** *A ring  $R$  is left hereditary if every submodule of a projective  $R$ -module is projective.*

Semisimple rings, domains and path algebras over a quiver are examples of left hereditary rings. A ring  $R$  is semisimple if the regular module  ${}_R R$  is a direct sum of simple submodules.

**Lemma 4.** *Let  $M$  be a projective module defined over a left hereditary ring  $R$ . Then for any submodule  $N$  of  $M$ ,*

$$\beta(N) \subseteq \langle E_N(0) \rangle \subseteq \beta_{co}(N).$$

**Proof:** Since  $M$  is projective, so is every submodule  $N$  of  $M$  by definition of a left hereditary ring.  $N$  projective, implies  $\beta(N) = \beta(R)N$ . Let  $m \in \beta(N)$ , then  $m = \sum_{i=1}^k a_i n_i$  where  $a_i \in \beta(R)$ ,  $n_i \in N$  and  $k \in \mathbb{N}$ . Since  $\beta(R)$  is nil,  $a_i n_i \in E_N(0) \subseteq \langle E_N(0) \rangle$ . This shows that  $\beta(N) \subseteq \langle E_N(0) \rangle$ . The second inequality follows from Lemma 1. ■

**Theorem 4.10.** *Let  $M$  be a projective module over a hereditary ring  $R$ . If a submodule  $N$  of  $M$  is 2-primal considered as a module, then  $N$  satisfies both the radical formula as well as the complete radical formula and hence*

$$\beta(N) = \langle E_N(0) \rangle = \beta_{co}(N).$$

**Proof:** If  $N$  is 2-primal (as a module), then  $\beta(N) = \beta_{co}(N)$ . Now apply Lemma 4 to get  $\beta(N) = \langle E_N(0) \rangle = \beta_{co}(N)$  and Lemma 3 to see that  $N$  satisfies both the radical formula as well as the complete radical formula. ■

**Corollary 4.11.** *Let  $M$  be a module over a semisimple ring  $R$ . If a submodule  $N$  of  $M$  is 2-primal considered as a module, then  $N$  satisfies both the radical formula as well as the complete radical formula.*

**Proof:** If  $R$  is a semisimple ring, then every  $R$ -module is projective. The rest follows from Theorem 4.10. ■

**Corollary 4.12.** *A semisimple commutative ring satisfies the radical formula.*

**Proof:** Since every module over a commutative ring  $R$  is 2-primal, applying Corollary 4.11 when  $R$  is semisimple shows that every submodule of an  $R$ -module satisfies the radical formula and hence  $R$  satisfies the radical formula. ■

**Corollary 4.13.** *A semisimple 2-primal ring satisfies the complete radical formula.*

**Proof:** A module  $M$  over a semisimple 2-primal ring  $R$  is projective and 2-primal by properties of semisimple rings and [5, Theorem 1] respectively. Since a semisimple ring is hereditary, every submodule  $N$  of such a module is also projective.  $N$  is 2-primal considered as a module by [5, Theorem 1]. By Theorem 4.10,  $N$  satisfies the complete radical formula. So,  $M$  satisfies the complete radical formula. ■

Corollaries 4.12 and 4.13 give us another situation where 2-primal rings behave like commutative rings.

**Proposition 8.** *The following statements are equivalent:*

1. *a semisimple 2-primal ring satisfies the complete radical formula,*
2. *a semisimple 2-primal ring satisfies the radical formula.*

**Proof:** Let  $R$  be a semisimple 2-primal ring. Then any  $R$ -module  $M$  is 2-primal. By Proposition 7,  $M$  satisfies the complete radical formula if and only if it satisfies the radical formula. ■

Example 3 shows that it is possible for a submodule to satisfy the complete radical formula when it neither satisfies the radical formula nor 2-primal.

**Example 3.** Let  $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$  where entries of matrices in  $M$  are from  $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$  and  $R = M_2(\mathbb{Z})$ . The zero submodule of the  $R$ -module  $M$  satisfies the complete radical formula, but  $M$  is neither 2-primal nor its zero submodule satisfies the radical formula.

**Proof:** It suffices to show that  $0 = \beta(M) \subsetneq \beta_{co}(M) = \langle E_M(0) \rangle = M$ . Let  $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$ ,

$$rM = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} a & a \\ c & c \end{pmatrix}, \begin{pmatrix} b & b \\ d & d \end{pmatrix}, \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \right\} \subseteq M$$

for any  $a, b, c, d \in \mathbb{Z}$ . There would be non-trivial proper submodules, namely;

$$N_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}, N_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\} \text{ and}$$

$N_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$  are not closed under multiplication by  $R$  since, for  $a$  and  $c$  odd,  $rN_1 \not\subseteq N_1$ , for  $b$  and  $d$  odd,  $rN_2 \not\subseteq N_2$  and for  $a$  odd but  $b, c, d$  even,  $rN_3 \not\subseteq N_3$ . This shows that  $M$  is simple and hence prime. So, we have  $\beta(M) = 0$ . However, if we take  $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in R$  and  $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$ ,  $am = 0$  but  $aM \neq 0$  since  $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq 0$ . This shows that  $M$  is not completely prime. So,  $M$  has no completely prime submodules, i.e.,  $\beta_{co}(M) = M$ . Note that

1.  $m_0 = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$
2.  $m_1 = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$  and  $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}^2 \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$
3.  $m_2 = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$  and  $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$
4.  $m_3 = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$  a linear combination of  $m_1$ .

This shows that  $\langle E_M(0) \rangle = M$ . ■

**Theorem 4.14.** *The following modules satisfy the complete radical formula:*

1. *a module with  $\beta_{co}(M) = 0$ , (e.g., when  $M$  is completely prime);*
2. *a module with  $\langle E_M(0) \rangle = M$  (e.g., a module given in Example 3);*
3. *the regular module  ${}_R R$  when  $R$  is 2-primal.*

*Moreover, a module with  $\beta_{co}(M) = 0$  (and the regular module  ${}_R R$  when  $R$  is 2-primal) is 2-primal, satisfies both the radical formula and a module analogue of Levitzki result for rings.*

**Proof:** For each the three modules,  $\beta_{co}(M) = \langle E_M(0) \rangle$ . Now, apply Lemma 3 to get the desired result. For the second part, since  $\beta(M) \subseteq \beta_{co}(M) = 0$  and  $\mathcal{N}_s(M) \subseteq \langle E_M(0) \rangle \subseteq \beta_{co}(M) = 0$ , we have  $\beta(M) = \mathcal{N}_s(M) = \langle E_M(0) \rangle = \beta_{co}(M)$ . For the regular modules  ${}_R R$ , apply Corollary 4.3 with the fact that  $\beta(R) = \beta({}_R R)$ ,  $\beta_{co}(R) = \beta_{co}({}_R R)$  and  $\mathcal{N}_s(R) = \mathcal{N}_s({}_R R)$ . ■

## 5 Application of modules that satisfy the complete radical formula

We know that a ring  $R$  is reduced if and only if it is a subdirect product of domains. In general, this structure theorem is not true in the module setting. This is due to the fact that not every completely semiprime submodule is an intersection of completely prime submodules. However, it holds when a module satisfies the complete radical formula, see Theorem 5.1.

**Definition 5.1.** A module  $M$  is a subdirect product of the modules  $S_\lambda$ ,  $\lambda \in \Lambda$  if there is an injective module homomorphism  $\sigma : M \rightarrow S = \prod_{\lambda \in \Lambda} S_\lambda$  such that  $\sigma \circ \pi_\lambda$  is surjective for all  $\lambda \in \Lambda$  and for every canonical surjection  $\pi_\lambda : S \rightarrow S_\lambda$ .

**Theorem 5.1.** Suppose that a module  $M$  satisfies the complete radical formula, then the following statements are equivalent:

1.  $M$  is completely semiprime,
2.  $\langle E_M(0) \rangle = 0$ ,
3.  $\beta_{co}(M) = 0$ ,
4.  $M$  is a subdirect product of completely prime modules.

**Proof:**

1  $\Leftrightarrow$  2. Follows from Proposition 5.

2  $\Leftrightarrow$  3. Since  $M$  satisfies the complete radical formula,  $\langle E_M(0) \rangle = \beta_{co}(M)$ . So,  $\langle E_M(0) \rangle = 0$  if and only if  $\beta_{co}(M) = 0$ .

3  $\Rightarrow$  4. Suppose that  $\beta_{co}(M) = 0$ . Let  $\{N_\lambda\}_{\lambda \in \Lambda}$  be a collection of all completely prime submodules of  $M$ . Then  $\cap_{\lambda \in \Lambda} N_\lambda = 0$  and  $M$  is a subdirect product of modules  $M/N_\lambda$ ,  $\lambda \in \Lambda$  which are completely prime. To see this, define  $\sigma : M \rightarrow \prod_{\lambda \in \Lambda} (M/N_\lambda)$  by  $\sigma(m) = (m + N_\lambda)_{\lambda \in \Lambda}$ . Then  $\text{Ker } \sigma = \cap_{\lambda \in \Lambda} \text{Ker } \pi_\lambda = \cap_{\lambda \in \Lambda} N_\lambda$ ,  $\sigma \circ \pi_\lambda$  is surjective for every  $\lambda \in \Lambda$  and  $\sigma$  is injective if and only if  $\cap_{\lambda \in \Lambda} N_\lambda = 0$ .

4  $\Rightarrow$  3. Let  $M$  be a subdirect product of completely prime modules  $\{S_\lambda\}_{\lambda \in \Lambda}$ , i.e., there is an injection  $\sigma : M \rightarrow \prod_{\lambda \in \Lambda} S_\lambda$  with  $\sigma \circ \pi_\lambda$  surjective, where  $\pi_\lambda : \prod_{\lambda \in \Lambda} S_\lambda \rightarrow S_\lambda$  is the canonical surjection. Then,  $\text{Ker}(\sigma \circ \pi_\lambda)$  is a completely prime submodule of  $M$ . Hence,  $\beta_{co}(M) \subseteq \bigcap_{\lambda \in \Lambda} \text{Ker}(\sigma \circ \pi_\lambda) = \text{Ker}(\sigma) = 0$  and  $\beta_{co}(M) = 0$ . ■

**Corollary 5.2.** *Let  $M$  be a module over a commutative ring. If  $M$  satisfies the radical formula, then the following statements are equivalent:*

1.  $M$  is semiprime,
2.  $\langle E_M(0) \rangle = 0$ ,
3.  $\beta(M) = 0$ ,
4.  $M$  is a subdirect product of prime modules.

**Proof:** For modules defined over commutative rings, prime (resp. semiprime) is indistinguishable from completely prime (resp. completely semiprime). Also, modules that satisfy the radical formula are indistinguishable from those that satisfy the complete radical formula. ■

**Corollary 5.3.** *If a module  $M$  satisfies the complete radical formula, then the submodule  $\beta_{co}(M)$  is the smallest completely semiprime submodule of  $M$ .*

**Proof:** If  $M$  satisfies the complete radical formula, then by Theorem 5.1, the zero submodule of  $M$  is a completely semiprime submodule of  $M$  if and only if  $\beta_{co}(M) = 0$ . In this case, there is no completely semiprime submodule of  $M$  smaller than  $\beta_{co}(M)$ . ■

**Corollary 5.4.** *Suppose that  $M$  is a module defined over a commutative ring. If  $M$  satisfies the radical formula, then the submodule  $\beta(M)$  is the smallest semiprime submodule of  $M$ .*

**Proof:** For modules over commutative rings,  $\beta(M) = \beta_{co}(M)$  and for a module  $M$  to satisfy the radical formula is equivalent to having  $M$  satisfy the complete radical formula. The notion of semiprime is indistinguishable from that of completely semiprime. ■

**Question 5.1.** *Can we have Corollary 5.2 for a module over a not necessarily commutative ring? i.e., a module to be semiprime if and only if it is a subdirect product of prime modules. Note that a not necessarily commutative ring is semiprime if and only if it is a subdirect product of prime rings.*

Whereas we do not know the answer, we hasten to mention that  $2 \Leftrightarrow 3 \Leftrightarrow 4 \Rightarrow 1$  is easy to prove. So, the question reduces to checking whether for modules  $M$  that satisfy the radical formula,  $M$  semiprime implies  $\beta(M) = 0$ . Note that, this is not true in general, see Example 3.



## 6 A module over a 2-primal ring is 2-primal

We have seen that a projective module over a 2-primal ring is 2-primal [5, Corollary 2.1], a finitely generated module (and hence a cyclic module) over a 2-primal ring is 2-primal [Theorem 4.9] and a module over a commutative ring (a commutative ring is 2-primal) is 2-primal. This further compels ones belief in the conjecture [5, Conjecture 2.1] which states that a module defined over a 2-primal ring is 2-primal. We show in Theorem 6.1 that this conjecture is true.

**Theorem 6.1.** *A module over a 2-primal ring is 2-primal.*

**Proof:** Let  $M$  be a module over a 2-primal ring  $R$ . We know that every module is a homomorphic image of a projective module. So, there exists a projective  $R$ -module  $P$  such that  $M$  is a homomorphic image of  $P$ . By [5, Corollary 2.1],  $P$  is a 2-primal module (since it is a projective module over a 2-primal ring). To complete the proof, it is enough to show that every homomorphic image of a 2-primal module is 2-primal. But this is easy to see since for every  $R$ -module epimorphism  $\phi : M \rightarrow M'$ , we have  $\phi(\beta(M)) = \beta(\phi(M)) = \beta(M')$  and  $\phi(\beta_{co}(M)) = \beta_{co}(\phi(M)) = \beta_{co}(M')$ . ■

Theorem 6.1 shows that 2-primal modules are abundant.

## 7 Questions in algebraic geometry

Much of the algebraic geometry is done using commutative rings. Naturally, one wonders whether the algebraic geometry already known for commutative rings can be developed for non-commutative rings. However, there are two challenges in trying to achieve this objective. 1) Unlike commutative rings, non-commutative rings have fewer ideals and hence fewer prime ideals. As such, there is not usually a good topological space that reflects the ideal structure and representation theory of a given ring. Hence, defining a projective scheme as a ringed topological space on the homogeneous primes of a ring would not be useful. 2) There isn't a good theory of localization for non-commutative rings. So, any attempt to develop a non-commutative algebraic geometry based on rings and their localizations will not work, see [10] and [22].

Against this background together with the behaviour of 2-primal rings having tendencies of commutative rings, some questions come to mind.

**Question 7.1.** *Do 2-primal rings have as many ideals (and hence as many prime ideals) as the commutative rings so that it is possible and useful to define a projective scheme as a ringed topological space on its homogeneous prime ideals?*

**Question 7.2.** *Can one develop a good theory of localization for 2-primal rings? In other words, is the theory of localization of 2-primal rings close to that of commutative rings*

that one can be able to do with 2-primal rings almost all that is done with commutative rings as regards localization?

An affirmative answer to any one of the two questions above will increase on the class of rings for which certain algebraic geometry can be done.

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